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On the singularity analysis of ordinary differential equations invariant under time translation and rescaling

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Abstract. The PT is applied to the general second- and third-order ordinary differential equations invariant under the two symmetries associated with time translation and rescaling in order to investigate their solvability and global integrability. The effect of the two symmetries on the compatibility conditions is determined and we show that, generally, these conditions are automatically a consequence of the resonance condition. Use is made of truncated Laurent series both in ascending and descending powers. As an example, the case of the generalized Chazy equation is presented.

1. Introduction

We have recently shown [8] that the singularity analysis of the Lotka Volterra and Quadratic Systems in two dimensions is intimately associated with the presence of first integrals. These systems possess in general one Lie point symmetry, that of invariance under time translation. This invariance, represented by the generator

$$G_1 = \frac{\partial}{\partial t} \tag{1}$$

is common in physical systems, in which the force law does not explicitly depend upon time. Another symmetry, common in nature, is that of self-similarity, which lies at the foundation of the dimensional analysis found so useful by engineers, and has the generator

$$G_2 = -qt\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}. \tag{2}$$

It is not unusual to find both symmetries in the same physical problem, for example the gravitational two-body problem. Since the two symmetries, (1) and (2), have the Lie bracket

$$[G_1, G_2] = G_1G_2 - G_2G_1 = -qG_1 \tag{3}$$

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and

$$G_1 \neq \rho(t,x)G_2 \tag{4}$$

the Lie algebra is that of Lie's type III of two-dimensional algebras [12] and, as such, represents the equivalence class of ordinary differential equations (ODEs) with two symmetries with properties (3) and (4). We consider it to be of interest to examine the properties of ordinary differential equations invariant under (1) and (2) from the viewpoint of singularity analysis. In particular we wish to consider second- and third-order ODEs with these invariances. The general form of a scalar second-order ordinary differential equation (SODE) invariant under the generators of time translation and rescaling, is

$$\ddot{x} + x^{(2q+1)}f(\xi) = 0 \tag{5}$$

and of the third-order ordinary differential equation (TODE) is

$$\dddot{x} + x^{(3q+1)}f(\xi,\eta) = 0 \tag{6}$$

where the overdot denotes $\mathrm{d}/\mathrm{d}t$, f is an arbitrary function and

$$\xi = \frac{\dot{x}}{x^{q+1}} \qquad \eta = \frac{\ddot{x}}{x^{2q+1}}. \tag{7}$$

Since (5) has two symmetries, it is integrable in the sense of being reducible to quadratures. This is very evident due to the existence of the first integral obtained from

$$\frac{\mathrm{d}x}{x} + \frac{\xi\,\mathrm{d}\xi}{(q+1)\xi^2 + f(\xi)} = 0. \tag{8}$$

In principle, as a consequence of the implicit function theorem, the integral can be inverted to give $\dot{x}$ in terms of x and the formal quadrature follows from

$$\mathrm{d}t = \frac{\mathrm{d}x}{\dot{x}(x)}. \tag{9}$$

In fact (8) allows a complete qualitative treatment of the equation and a quantitative asymptotic solution. This possibility of a formal quadrature as given by (9) does not apply to (6).

The regular Painlevé analysis (PA) is based on the identification of the singularities and expansion of the function as a Laurent series in ascending powers of $t - t_0$, which we call 'right Painlevé series' (RPS), where the location of the singularity is at t_0. We are concerned with the behaviour of the function as $t \mapsto t_0$. In the case of an autonomous equation, i.e. one invariant under time translation, t_0 is a function only of the initial conditions if it is a movable singularity. More recently there has been interest [11, 6] in doing the Laurent expansion in descending powers of $t - t_0$, which we call 'left Painlevé series' (LPS).

A standard technique used when a symmetry is present is the reduction of the order of the ODE to one of lower order. When the equation is invariant under time translation, the dependent variable becomes the independent variable of the reduced equation. Thus an RPS in $t - t_0$ for the original equation must necessarily become an LPS in $x - x_0$ at the lower order. In fact, when one realizes this duality, to distinguish between the RPS and the LPS does not seem to be consistent with the concept of using Laurent expansions as solutions of ODEs. We should note here that the existence of the symmetries G_1, G_2 and the passing of the Painlevé test (PT) are not one to one. Thus an equation may be invariant under time translation and rescaling and yet not pass the PT (see the example treated in section 10). However, as we shall see below, invariance under rescaling does mean that there can exist a singularity when all terms in an equation are dominant with interesting consequences on the structure of the Painlevé series.

We wish to make clear the connection between the content of this paper and the standard PT. Our procedure, as was clearly demonstrated in [8], is to construct a Laurent series without making the distinction between resonance and compatibility, i.e. we explicitly calculate all of the terms of the series and remain sensitive to the definition of a Laurent series. The leading-order analysis of an equation possessing the self-similar property gives a self-similar solution by the nature of its construction which is identical to the search for a self-similar symmetry. It may also give other solutions from a subset of all of the terms in the equation. In this case the terms considered must be separately homogeneous in the independent and dependent variables. These self-similar solutions are known often to be asymptotic solutions.

In this work we are more concerned with the leading-order behaviour of solutions, but we must keep the PT in mind as this was the initial inspiration for our investigations. We must, however, emphasize that our approach does differ from that of the PT. This is because the greater generality of the self-similar symmetry draws us away from the analytic properties of the PT in the complex plane. Nevertheless, we do consider the implication of the number of arbitrary constants which can appear in the Painlevé series and its relationship to the space of initial conditions. Suppose that we have a number of arbitrary constants in the Painlevé series equal to the order of the equation. Then we can span the whole of the space of initial conditions and have a series representation of the general solution. On the other hand an insufficient number of arbitrary constants means that the space of initial conditions cannot be spanned and so the Painlevé series represents only a partial solution.

When we have the requisite number of arbitrary constants, we have hightened expectations of being able to obtain an explicit solution. Thus for a SODE we shall show that there exists a reduction of the solution to quadratures. In practice it may even be something better. For example, Bouquet *et al* [2] and Lemmer and Leach [11] found that, although $\ddot{x} + x\dot{x} + kx^3 = 0$ is always integrable in the sense of reduction to quadratures, the integration can be explicitly performed when the parameter k is such that the equation possesses the weak PT. In the case of a TODE we can expect a third symmetry which will enable reduction of order of the equation to a 'nice' quadrature. We should not expect this symmetry to be of point or contact type. It may equally well be nonlocal, but of the useful variety [7].

In entering our investigations we definitely leave aside the search for the passing of the strict PT for a more operational and empirical approach in the interests of furthering understanding and finding generalizations. In this paper we direct attention to the application of the singularity analysis, as well as the Painlevé test, to (5) and (6) and the use of both LPS and RPS in establishing their integrability properties.

The paper is organized as follows: in section 2 we discuss different types of singularities and asymptotic behaviours which can occur. In section 3 we give a general expression to the nth-order equation satisfying the two mentioned symmetries. The PT of the SODE is analysed in section 4. In section 5 we make two comments on the SODE imbedding the PT in a more general form. Then, the PT of the TODE is considered in section 6 and a general property of the LPS is outlined for the nth-order equation satisfying the two symmetries mentioned in section 7. Then in section 8, we consider the reduced equations obtained by taking into account the time translation symmetry. In section 9 we are concerned with the algebraic nature of the first integral of the SODE and finally in sections 10 and 11 we apply the results of the preceding sections to characteristic examples.

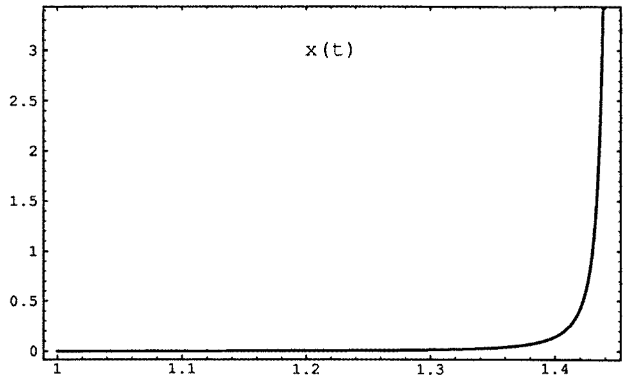

Figure 1. Curve $x(t)$ of equation (10) with $k = -\frac{3}{2}$, for a start at $x(0) = 8$, $x'(0) = 16$ and $x''(0) = 48$. The scale factor of x is 10^5.

2. Different types of singularities and asymptotic behaviour

We seek to imbed the PT into the self-similar structure. This even admits the possibility of different kinds of self-similar behaviour. For example, in the leading-order analysis of the generalized Chazy equation, (see section 10) two possible types of leading-order behaviour are identified. When all three terms in

$$\dddot{x} + x\ddot{x} + k\dot{x}^2 = 0 \tag{10}$$

are considered, we obtain the leading order

$$x = \frac{6}{(k+2)(t-t_0)}$$

which is the self-similar solution. Another choice for the leading terms is possible by taking only the second and third term (since the second and the third terms rescale in the same way, taking the first term obliges us to take all three). In that case we obtain $x = a(t-t_0)^{1/(1+k)}$, where a is an arbitrary constant, which corresponds to the self-similar solution of the equation without the first term. Which solution, the one from the three terms or the one from the two terms, will dominate the asymptotic behaviour is determined by the initial conditions for a given value of the parameter k. The form of the second asymptotic solution suggests immediately that $k = -1$ is a critical value since $k < -1$ introduces a possible blow-up of the solution at a finite time while $k > -1$ implies x and t going to infinity at least for some initial values. This is illustrated in figures 1 and 2 respectively.

Let us consider now the example $f(\xi) = \lambda\xi^2 + \mu\xi + \nu$, corresponding to the equation

$$\ddot{x} + \lambda\frac{\dot{x}^2}{x} + \mu x^q\dot{x} + \nu x^{2q+1} = 0. \tag{11}$$

The invariance of (11) under the homothetic transformation $t = \alpha\bar{t}$ and $x = \beta\bar{x}$ shows that we have two essential parameters λ and ν/μ^2. Without any loss of generality we take $\mu = 1$. Then (8) is

$$\frac{\mathrm{d}x}{x} + \Phi(\xi)\,\mathrm{d}\xi = 0 \tag{12}$$

with

$$\Phi(\xi) = \frac{\xi}{(q+\lambda+1)\xi^2 + \xi + \nu}.$$

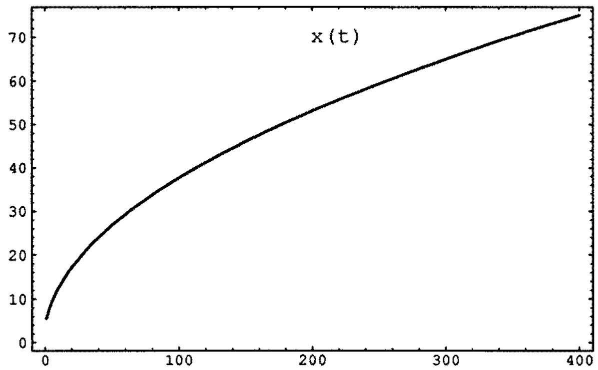

Figure 2. Curve $x(t)$ of equation (10) with $k = 1$, for a start at $x(0) = 4$, $x'(0) = 2$ and $x''(0) = -1$.

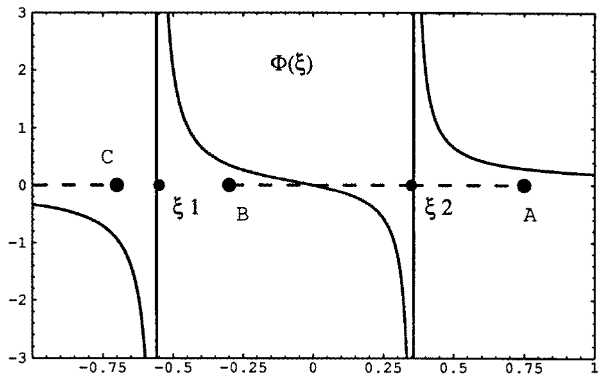

Figure 3. Curve $\Phi(\xi)$ of equation (11) in the case $q = 2$, $\lambda = 2$, $\mu = 1$ and $\nu = -1$.

Let us suppose $q + \lambda + 1 > 0$ and ν such that we have two real roots for the equation $(q+\lambda+1)\xi^2+\xi+\nu = 0$. For example, if $\lambda = 2$, $\nu = -1$ and $q = 2$, we have one negative and one positive root $\xi_1 = -0.5582$ and $\xi_2 = 0.3582$. Let us start with $x(0) = 1$ (point A on figure 3). This is possible since the above mentioned rescaling imposes $\beta^q\alpha\mu = 1$ which leaves the possibility of an arbitrary β and a subsequent rescaling to one of x_0. Since $\xi(0) > \xi_2$, $\dot{x} > 0$ and, finally, $\mathrm{d}x/x > 0$. Now from (12) $\Phi(\xi)\,\mathrm{d}\xi < 0$ and as $\Phi(\xi) > 0$, one obtains $\mathrm{d}\xi < 0$. Consequently ξ reaches the value $\xi = \xi_2$ with

$$\mathrm{d}x\, x^{-(q+1)} = \xi_2\,\mathrm{d}t.$$

This last equation gives

$$x = (-\xi_2 q)^{-1/q}(t - t_0)^{-1/q}$$

which, according to the sign of t_0, indicates either an explosion of the solution in a finite time t_0 (if $t_0 > 0$) or a decrease to zero as $t \mapsto \infty$ in $t^{-1/q}$ (if $t_0 < 0$). Here, since ξ is always positive, x must increase and we have an explosion in a finite time. In fact as in our example $q = 2$ and $\xi_2 > 0$ we have $(-\xi_2 q)^{-1/q}$ pure imaginary. Since x must be real, $(t - t_0)^{-1/q}$ must also be purely imaginary indicating that $t_0 > 0$ and the subsequent explosion at $t = t_0$. The same kind of argument shows easily that any starting point

$$x(0) > 0 \qquad \xi_1 < \xi(0) < \xi_2$$

(point B on figure 3) leads to an increase in ξ which passes the point $\xi = 0$ and again reaches ξ_2 with an explosion of x at a finite time. The last case deals with the trajectory of a point initially at $\xi(0) < \xi_1$ (point C on figure 3). It is easily shown that $\xi \mapsto -\infty$, indicating that ξ_1 is repulsive and $x \mapsto 0$. However, the asymptotic form can be obtained by noting that, if $\xi \mapsto -\infty$, (12) takes a simple form and can be integrated with

$$\frac{\mathrm{d}x}{x} + \frac{\mathrm{d}\xi}{(q+\lambda+1)\xi} = 0. \tag{13}$$

The relation $x\xi^{1/(q+\lambda+1)} = c$ where c is an arbitrary constant, leads to $\dot{x}x^{\lambda} = c^{q+\lambda+1}$ and finally to the relation

$$x = a(t-t_0)^{\frac{1}{\lambda+1}} \tag{14}$$

where a is an arbitrary constant. Note that (14) is the solution of (11) in which we take into account only the two first terms. Taking for $x(t)$ the expression given by (14) we see that the neglected terms in (11), namely $x^q\dot{x}$ and x^{2q+1}, when divided by $\ddot{x}$ vary respectively as

$$(t-t_0)^{(q+\lambda+1)/(\lambda+1)} \qquad \text{and} \qquad (t-t_0)^{2(q+\lambda+1)/(\lambda+1)}. \tag{15}$$

If the exponents of $t - t_0$ in (15) are positive, it is possible indeed to neglect the last two terms of (11). This indicates that the critical values are $\lambda = -(q+1)$ and $\lambda = -1$.

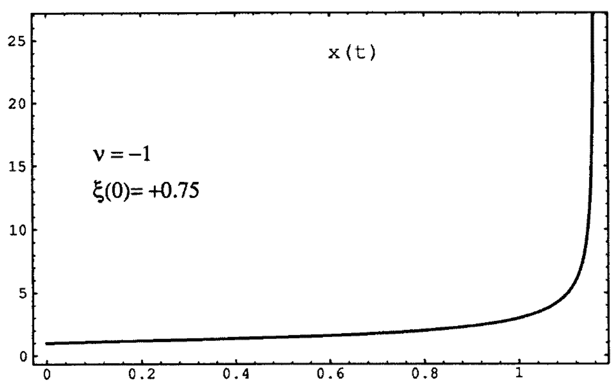

Figure 4. Curve $x(t)$ for a start at point A of figure 3.

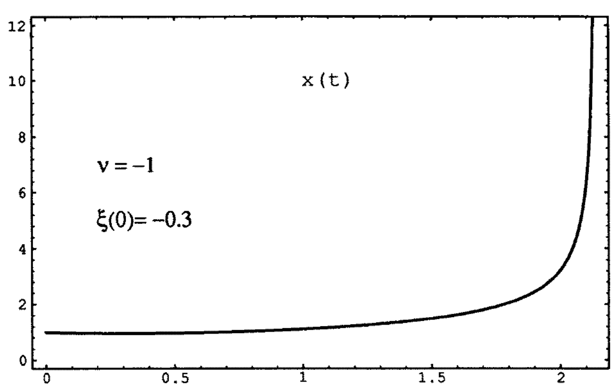

Figure 5. Curve $x(t)$ for a start at point B of figure 3.

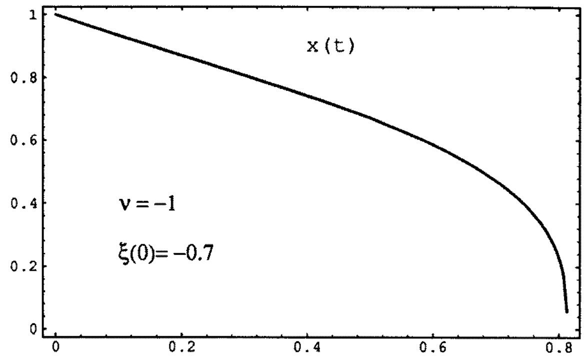

Figure 6. Curve $x(t)$ for a start at point C of figure 3.

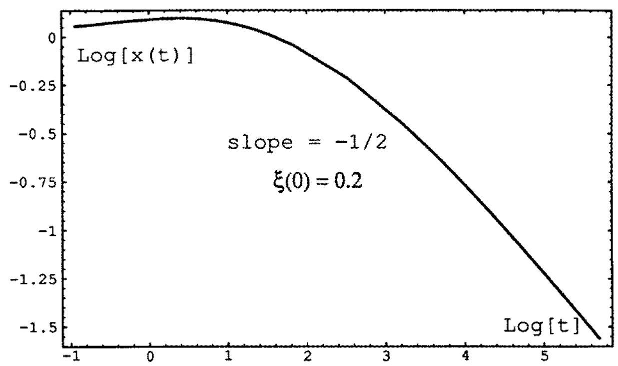

Figure 7. Curve $x(t)$ of equation (11) for a start at $\xi(0) > 0$ and $x(0) = 1$ in the case $q = 2$, $\lambda = 2$, $\mu = 1$ and $\nu = 0.03$.

Figures 4–6 exhibit the curves $x(t)$ respectively for a start at points A, B and C of figure 3 in the case of the above mentioned values $q = 2$, $\lambda = 2$, $\mu = 1$ and $\nu = -1$. Indeed we recover the two explosions and the cancellation of $x(t)$ in the third case as $(t - t_0)^{1/3}$. Figure 7 is for the values $q = 2$, $\lambda = 2$, $\mu = 1$ and $\nu = 0.03$ with the two roots for ξ negative. In that case starting with a positive $\xi(0)$ and $x(0) = 1$, x first increases and then decreases to zero as $t^{-1/q}$when $t \mapsto \infty$. For some problems the leading order corresponds to an explosion as $(t - t_0)^{-1/q}$ when $t \mapsto t_0$, for others to a decrease as $t^{-1/q}$ when $t \mapsto \infty$.

3. On the general nth-order differential equation

In this section we give a general expression to the nth-order differential equations possessing the symmetries of time translation and self-similarity.

Proposition 1. Any nth-order ordinary differential equation (NODE) of the form

$$x^{(n)} + g(x, \dot{x}, \ldots, x^{(n-1)}, t) = 0 \tag{16}$$

having the two symmetries of time translation and self-similarity, can be expressed as

$$x^{(n)} + x^{nq+1} f\left(\frac{\dot{x}}{x^{q+1}}, \ldots, \frac{x^{(n-1)}}{x^{(n-1)q+1}}\right) = 0. \tag{17}$$

Proof. Let us employ first the symmetry characterizing the invariance under the time translation. Then, under the action of G_1, (16) becomes

$$x^{(n)} + g(x, \dot{x}, \ldots, x^{(n-1)}) = 0. \tag{18}$$

Now apply G_2 which represents the self-similarity. For that we compute the nth extension of G_2, namely

$$G_2^{[n]} = -qt\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + (q+1)\dot{x}\frac{\partial}{\partial \dot{x}} + \cdots + (nq+1)x^{(n)}\frac{\partial}{\partial x^{(n)}}.$$

The action of $G_2^{[n]}$ on (16) induces the partial differential equation

$$x\frac{\partial g}{\partial x} + (q+1)\dot{x}\frac{\partial g}{\partial \dot{x}} + \cdots + [(n-1)q+1]x^{(n-1)}\frac{\partial g}{\partial x^{(n)}} - (nq+1)g = 0$$

which we solve by the method of the characteristics. The associated Lagrange's system is

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}\dot{x}}{(q+1)\dot{x}} = \cdots = \frac{\mathrm{d}x^{(n-1)}}{[(n-1)q+1]x^{(n-1)}} = \frac{\mathrm{d}g}{(nq+1)g}$$

and the characteristics are

$$\xi_j = \frac{x^{(j)}}{x^{jq+1}} \qquad (j = 1, 2, \ldots, n-1).$$

Consequently the solution of this linear partial differential equation is

$$g = x^{(nq+1)}f(\xi_1, \ldots, \xi_{n-1})$$

where f is an arbitrary function of ξ_j $(j = 1, \ldots, n-1)$ and finally

$$x^{(n)} + g(x, \ldots, x^{(n-1)}) = 0 \iff x^{(n)} + x^{nq+1}f(\xi_1, \ldots, \xi_{n-1}) = 0.$$

□

4. The PT for the SODE

In this section we apply the PT to the SODE. The PT is based on the Laurent series which we recall, has the general form

$$\sum_{j=-\infty}^{+\infty} a_j\tau^j$$

where $\tau = t - t_0$ and t_0 is the location of the movable singularity. The conventional PT leads to an RPS of the form $\sum_{j=0}^{j=\infty} a_j\tau^{j+p}$, but there is no reason not to consider a LPS of the form $\sum_{j=-\infty}^{j=0} a_j\tau^{j+p}$ with $p < 0$ for the RPS and no restriction on p for the LPS. In the LPS τ may be taken as t without any loss of generality. This is because the series represents an expansion outside a disk centred on t_0 and, when one is interested in the behaviour as $t \mapsto \infty$, the finite value of t_0 loses relevance. This interpretation of the LPS is in agreement with standard treatments of Laurent expansions. In (5) and (6) all terms have the same weight due to the presence of the self-similar property. For both RPS and LPS the leading order is in $\tau^{-1/q}$.

We want to use the PT to obtain a criterion of full integrability, i.e. we must have, in the series, as many arbitrary coefficients as the order of the equation. We have already noted that, for an RPS, t_0 provides one of these arbitrary coefficients. Consequently, we need one coefficient in the case of a SODE and two in the case of a TODE. For an LPS we see that, since we expand for $t \to \infty$, we lose the arbitrary t_0 (which located the movable singularity). Hence we need, respectively, two and three arbitrary coefficients in the LPS of SODE and TODE. We can formulate the following propositions

Proposition 2. For a SODE to pass the Painlevé test, i.e. for an RPS to have a_r arbitrary—which with t_0 provides the two necessary arbitrary coefficients—and for an LPS to have again a_r arbitrary (now r is negative)—which together with a_{-1} gives the two arbitrary coefficients—we need only the relation $f_\xi + [2(q+1) - qr]\xi_0 = 0$ to be fulfilled for r positive (RPS) or negative (LPS) without any additional relation.

Proof. Let us consider first the case of an RPS for the SODE. Taking (7) into account, (5) is now written

$$\eta + f(\xi) = 0. \tag{19}$$

For the leading term $a_0\tau^{-1/q}$ it is easily checked that ξ and η go to two constants. We obtain

$$\xi \to \xi_0 = -\frac{1}{qa_0^q} \qquad \eta \to \eta_0 = \frac{q+1}{q^2a_0^{2q}}. \tag{20}$$

Then $\eta_0 = (q+1)\xi_0^2$, while (19) becomes the algebraic equation

$$f(\xi_0) + (q+1)\xi_0^2 = 0 \tag{21}$$

giving the value of ξ_0. We may have many values of ξ_0 and, consequently, the analysis given below should be applied to all values. We will return to the coherence of the different possible solutions for ξ_0 below. Let us now search for one arbitrary coefficient in the RPS. We write

$$x(t) = a_0\tau^{-1/q} + a_r\tau^{r-1/q} \tag{22}$$

where r is an integer. We should begin with $r = 1$ since, as we have already pointed out, we seek not only the resonances as in the ARS algorithm but the possibility of getting a_r arbitrary. Introducing (22) in (19) and keeping the next term of the Taylor expansion in τ we get

$$\eta_0 + \eta_r + f(\xi_0 + \xi_r) = 0$$

where ξ_0 and η_0 satisfy (21), namely $\eta_0 + f(\xi_0) = 0$, and ξ_r and η_r are infinitesimal in τ^r. We must have

$$\eta_r + \xi_r f_\xi = 0 \tag{23}$$

where the derivative $f_\xi = \mathrm{d}f/\mathrm{d}\xi$ is taken at point $\xi = \xi_0$. A little algebra shows that (23) can be written

$$(r+1)\{f_\xi + [2(q+1) - qr]\xi_0\}a_r\tau^r = 0. \tag{24}$$

The case $r = -1$ is not relevant since we are building an RPS. Now either

$$f_\xi + [2(q+1) - qr]\xi_0 = 0 \tag{25}$$

and a_r can be arbitrarily selected and the equation passes the PT or $a_r = 0$. At this point we must remark that in general the equation involving a_r is of the form $Aa_r + B = 0$, where A and B are algebraic expressions depending on r and the parameters of the equations (see for example [8]). Hence, if we want a_r to be arbitrary, we need $A = 0$ (resonance condition) and $B = 0$ (compatibility condition). The interesting point of (24) is that B is not present. So there is no need to look for a compatibility condition as it is automatically satisfied. This is connected to the fact that all terms are dominant in the SODE. It is only when the similarity symmetry is broken that compatibility conditions must be explicity satisfied. We return to this point when we consider the construction of the LPS and RPS of both SODEs and TODEs. Therefore, in the case of equations possessing the two symmetries we have just one relation (25) to be fulfilled for each ξ_0 which, of course, agrees with the resonance condition of the ARS algorithm and all terms are zero up to the first integer r which fulfills relation (25).

If there is no integer solution for r in (24), all coefficients are zero. Of course this does not mean that around the singularity there is no correction to the leading-order term, but because we have taken a Laurent series, if this correction term is not in τ^{r+p}, the algorithm will not detect it (see next paragraph).

Now let us apply an LPS to the SODE. At first we examine the obtaining of two of the first three coefficients a_{-1}, a_{-2}, a_{-3} as arbitrary. We consider the term next to the leading term

$$x = t^{-1/q}\left(a_0 + \frac{a_{-1}}{t}\right).$$

As before, we compute ξ_{-1}, η_{-1}, the t^{-1} terms in the expansion of ξ and η finding that $\xi_{-1} = \eta_{-1} = 0$ for any value of a_{-1}. This occurs generally when the equation possesses the self-similar property as will be seen later.

Afterwards we build, by recurrence, the second arbitrary coefficient. The first step is to obtain a_{-2}. We introduce the asymptotic expansion in t^{-2} of η, ξ into (19) and find that

$$\eta_0 + \eta_{-1} + \eta_{-2} + f(\xi_0) + (\xi_{-1} + \xi_{-2})f_\xi + \tfrac{1}{2}\xi_{-1}^2 f_{\xi\xi} + \tfrac{1}{2}\eta_{-1}^2 f_{\eta\eta} + \xi_{-1}\eta_{-1}f_{\xi\eta} = 0$$

and, because of the relations $\eta_0 + f(\xi_0) = 0$ and $\xi_{-1} = \eta_{-1} = 0$, we can write

$$\eta_{-2} + \xi_{-2}f_\xi = 0$$

with

$$\begin{aligned}\xi_{-2} &= -a_0^{-2-q}[2a_0a_{-2} - (q+1)a_{-1}^2]/2 \\ \eta_{-2} &= a_0^{-2q-2}(2q+1)[2a_0a_{-2} - (q+1)a_{-1}^2]/q\end{aligned} \tag{26}$$

and

$$\eta_{-2} + \xi_{-2}f_\xi = 0 \iff [2a_0a_{-2} - (q+1)a_{-1}^2][f_\xi + 2(2q+1)\xi_0] = 0.$$

Now either $f_\xi + 2(2q+1)\xi_0 = 0$ or $2a_0a_{-2} - (q+1)a_{-1}^2 = 0$. In the first case a_{-2} is arbitrary and we can stop here since we have our two arbitrary coefficients a_{-1} and a_{-2}. In the second case $a_{-2} = (q+1)a_{-1}^2/(2a_0)$ not arbitrary we must proceed to the following step. However, first we make the fundamental observation that this is possible because of the structure of ξ_{-2} and η_{-2} as given by (26) which leads to $\xi_{-2} = \eta_{-2} = 0$. For this next step (i.e. $r = -3$) we perform a Taylor expansion in powers of t^{-3} in (19) where we take into account

$$\eta_{-1} = \xi_{-1} = \eta_{-2} = \xi_{-2} = 0.$$

Most of the terms disappear in the t^{-3} term of the Taylor expansion and we finally obtain

$$\eta_{-3} + \xi_{-3}f_\xi = 0 \tag{27}$$

where

$$\xi_{-3} = \frac{-a_0^{-q-3}}{3}[a_{-2}^3(q^2+3q+2) + 6a_0^2a_{-3} - 6a_0a_{-1}a_{-2}(q+1)]$$

$$\begin{aligned}\eta_{-3} = \frac{2a_0^{-2q-3}}{3q}&[2a_{-2}^3(2q^3+5q^2+4q+1) + a_0^2a_{-3}(15q+6) \\ &-a_0a_{-1}a_{-2}(18q^2+21q+6)]\end{aligned}$$

which with $a_{-2} = (q+1)a_{-1}^2/(2a_0)$ becomes

$$\begin{aligned} \xi_{-3} &= -a_0^{-q-3}[-a_{-1}^3 + 6a_0^2 a_{-3} - 3a_{-1}^3 q - 2a_{-1}^3 q^2]/3 \\ \eta_{-3} &= (5q+2)a_0^{-2q-3}[-a_{-1}^3 + 6a_0^2 a_{-3} - 3a_{-1}^3 q - 2a_{-1}^3 q^2]/(3q). \end{aligned} \tag{28}$$

Taking (28) into account in (27) we obtain finally

$$[-a_{-1}^3 + 6a_0^2 a_{-3} - 3a_{-1}^3 q - 2a_{-1}^3 q^2][f_\xi + (5q+2)\xi_0] = 0.$$

We see the emergence of the same process found at the preceding step: either $f_\xi + (5q + 2)\xi_0 = 0$, i.e. a_{-3} is arbitrary and we can stop, or $-a_{-1}^3 + 6a_0^2 a_{-3} - 3a_{-1}^3 q - 2a_{-1}^3 q^2 = 0$ and $a_{-3} = (q+1)(2q+1)a_{-1}^3/(6a_0^2)$ and we have $\xi_{-3} = \eta_{-3} = 0$. Note that the two relations obtained for the respective arbitrariness of a_{-2} and a_{-3}, namely $f_\xi + (4q+2)\xi_0 = 0$ and $f_\xi + (5q+2)\xi_0 = 0$, are obtained by taking $r = -2$ and $r = -3$ in (24) which is the necessary relation to have a_r arbitrary in an RPS.

Now we conclude assuming that up to a rank $i = r+1$ (remember that r is negative), we have $\xi_{-i} = \eta_{-i} = 0$. In that case the very complicated nature of the equation, collecting the terms for the coefficient of t^r, which *a priori* involves all the derivatives up to the order $-r$, simplifies and reduces to

$$\eta_r + \xi_r f_\xi = 0$$

where ξ_r and η_r are given by

$$\begin{aligned} \xi_r &= \frac{a_0^{r-q}}{r}\{|r|!a_r a_0^{-(r+1)} - (q+1)\cdots[1-(r+1)q]a_{-1}^{-r}\} \\ \eta_r &= -\frac{2(q+1)-qr}{qr} a_0^{r-2q}\{|r|!a_r a_0^{-(r+1)} - (q+1)\cdots[1-(r+1))q]a_{-1}^{-r}\} \end{aligned}$$

and we recognize that again ξ_r and η_r have a common factor containing a_0 and the arbitrary constant a_{-1}. Now either this common factor cancels (and we have $\xi_r = \eta_r = 0$) or the coefficient in front is zero and we have

$$f_\xi + [2(q+1) - qr]\xi_0 = 0$$

thereby generalizing the relation $f_\xi + (5q+2)\xi_0 = 0$ obtained previously for the case $r = -3$. Consequently the relation needed to pass the PT (24) is the same for an RPS or an LPS, proving the proposition. □

5. Comments on the PT

At this point, two comments are in order. The first deals with the expansion of the solution around the singularity and shows how, for an RPS, the PT selects the values of the parameters leading to a Laurent series. The second deals with the building of an LPS when the self-similarity property is not present and how, in this case, we must introduce the begining of a Psi series.

5.1.

Anticipating an example which will be treated extensively in section 10, we want to show how the PT picks up the values of the parameters for which the expansion around the singularity introduces terms in $\tau^{-1/q+r}$ where r is an integer. Consider the equation

$$\ddot{x} + x\dot{x} + kx^3 = 0 \tag{29}$$

which corresponds to $q = 1$, $f(\xi) = \xi + k$ in (5). Let us search an LPS. Then, except for a set of k of null measure, (i.e. except for special values of k) all the coefficients are zero, while obviously a series giving the correction to the singularity does exist. To find it we introduce the function $g = g(\tau)$ as

$$x = A/\tau + g \tag{30}$$

where A/τ is the leading term with A given by

$$kA^2 - A + 2 = 0. \tag{31}$$

We introduce (30) in (29) to obtain the equation for g and consider the linearized version of this equation, in the sense that $g(\tau) \to 0$ when $\tau \to 0$. A little algebra gives the linearized g equation

$$\ddot{g} + A\dot{g}/\tau + (2A - 6)g/\tau^2 = 0. \tag{32}$$

This is a Euler-type equation, the solution of which is given by

$$g = K_1\tau^{\lambda_1} + K_2\tau^{\lambda_2} \tag{33}$$

where λ_1 and λ_2 are the roots of the equation

$$\lambda(\lambda - 1) + A\lambda + (2A - 6) = 0. \tag{34}$$

The roots are $\lambda = -2$ and $\lambda = 3 - A$ which correspond to an expansion

$$x = \tau^{-1}\left(A + \frac{K_1}{\tau} + K_2\tau^{4-A} + \cdots\right).$$

The term in K_1/τ in the parentheses corresponds to a resonance at $r = -1$ and must be disregarded since we deal with an RPS. The term in τ^{4-A} corresponds to a resonance $r = 4 - A$ if A is an integer. But selecting the values of k such that A is an integer is exactly what does the PT. To see it we note that (25) gives $\xi_0 = -1/(4 - r)$ since $f_\xi = 1$ and $q = 1$. Remembering that $\xi_0 = -1/A$ (from (20) with $q = 1$) we see that $A = 4 - r$ and the set of k values will be obtained in the PT by introducing $A = 4 - r$ in (31) with the result

$$k = \frac{2 - r}{(4 - r)^2} \tag{35}$$

which can be generalized to values of $q \neq 1$ (see section 10). The important thing is that we have, for a given k, a correction to the leading order which is not an integer power of τ except if the PT is fulfilled. Since the algorithm building the Painlevé series supposes an expansion in integer powers, it cannot pick up the cases where A is not an integer and misses the term in τ^{4-A}.

5.2.

A comment is in order concerning the ARS resonance at $r = -1$. This resonance does not mean automatically that a_{-1} in an LPS is arbitrary. In fact this property is not true if the equation does not possess the self-similar property, which happens when there is a symmetry breaking term in the equation. For example, consider the equation

$$\ddot{x} + x\dot{x} + kx^s = 0$$

with $s \neq 3$. We cannot any more have a balance involving the three terms of the equation. We take first $s = 2$. Then the only possible balance is between the first and third term

(Remember we have $t \to \infty$ since we are seeking an LPS). The leading term is now $x = A/t^2$ with A given by

$$6 + Ak = 0. \tag{36}$$

Now writing the usual LPS with $x = A/t^2 + B/t^3$ (to find the usual resonance at $r = -1$) brings

$$B(6 + Ak) = A^2 \tag{37}$$

which is impossible taking (36) into account. This shows that the arbitrariness of a_{-1} (and the subsequent possibility to buid an LPS) was a consequence of the self-similarity property of the system. To find the nature of the series in t^{-1} we write

$$x = \frac{A}{t^2} + g. \tag{38}$$

We get $\ddot{x} = 6A/t^4 + \ddot{g}$; $x\dot{x} = (-2A^2)/t^5 - 2Ag/t^3 + \dot{g}A/t^2 + g\dot{g}$; $kx^2 = kA^2/t^4 + 2Akg/t^2 + kg^2$. Then we neglect the terms in $g\dot{g}$ and kg^2 and also the term $-2Ag/t^3$ obviously negligible compared with $2Akg/t^2$. So the equation for g writes

$$\ddot{g} + \frac{A}{t}\dot{g} + \frac{2Ak}{t^2}g - \frac{2A^2}{t^5} = 0. \tag{39}$$

To try $g = B/t^3$ is useless and brings back (37) (which is impossible). We try for g the following expansion

$$g = \frac{B\ln t}{t^3} + \frac{C}{t^3}. \tag{40}$$

Introducing (40) in (39) and keeping the terms up to $\ln t/t^5$ and $1/t^5$ we get

$$t^{-5}[2B(6 + Ak)\ln t - 2A^2 - 7B + 2C(6 + Ak)] = 0. \tag{41}$$

Taking (36) into account we recognize that we must take $B = -2A^2/7$ while C is arbitrary. Similar results can be obtained with $s = 4$ with, now, a balance between $\ddot{x}$ and $x\dot{x}$. The leading term is $x = 2/t$ and we seek an expansion

$$x = 2/t + B\ln t/t^2 + C/t^2$$

and we find again that C can be arbitrarily chosen while $B = 16k/3$.

We note, consequently, that for an equation not possessing the similarity symmetry, an LPS does not seem possible but instead a logarithmic term—indicating the begining of a Psi series—must be introduced. The implication for the possibility of partial integrability is not yet understood but the presence of a logarithmic term destroys the possibility of analytic solutions. Now we come back to equations possessing the two symmetries.

6. The PT for the TODE

Proposition 3. For a TODE to pass the Painlevé test, i.e. for an RPS to have two arbitrary a_r (a_{r_1} and a_{r_2}), the ARS resonance condition is fulfiled for r_1 and r_2. No other conditions are needed if $r_1 < r_2 < 2r_1$, otherwise a further condition must be fulfilled. For an LPS to have the three first a_r arbitrary, only the ARS resonance conditions are needed.

Proof. Introducing

$$\zeta = \frac{\dddot{x}}{x^{3q+1}} \tag{42}$$

in (6), the TODE writes

$$\zeta + f(\xi, \eta) = 0. \tag{43}$$

For the leading term (43) becomes an algebraic equation in ξ_0, η_0 and ζ_0 with the same relation (20) between a_0 and ξ_0. As before, $\eta_0 = (q+1)\xi_0^2$ and here $\zeta_0 = (q+1)(2q+1)\xi_0^3$. Consequently the algebraic equation giving ξ_0 (and then a_0) is

$$f(\xi_0, (q+1)\xi_0^2) + (q+1)(2q+1)\xi_0^3 = 0. \tag{44}$$

To pass the PT we need two arbitrary coefficients in the RPS. Although the coefficients a_s are zero for $1 \leqslant s < r$, where a_r is the first arbitrary coefficient as in the SODE, this is not necessarily true for $s > r$. Now for $s > r$ (24) is no longer automatically homogeneous and we must check what further relations are needed. These are the so-called compatibility conditions. Again we shall see that the self-similarity property brings a great simplification. To see what new relations are found we consider an RPS and, for the sake of simplicity, confine our attention to the three first terms (plus a_0) i.e.

$$x(t) = \tau^{-1/q}(a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3).$$

Again we expand ξ, η, ζ in powers of τ with

$$(\xi, \eta, \zeta) = \sum_{j=0}^{3}(\xi_j, \eta_j, \zeta_j)$$

where the terms are in τ^j with $j = 0, 1, 2, 3$. Then (43) can be divided into the following three equations, after introduction of the expansion of ξ, η, ζ and gathering terms of the same degree in τ

$$\zeta_1 + \xi_1 f_\xi + \eta_1 f_\eta = 0 \tag{45}$$

$$\zeta_2 + \xi_2 f_\xi + \eta_2 f_\eta + \tfrac{1}{2}\xi_1^2 f_{\xi\xi} + \tfrac{1}{2}\eta_1^2 f_{\eta\eta} + \xi_1\eta_1 f_{\xi\eta} = 0 \tag{46}$$

$$\begin{aligned}&\zeta_3 + \xi_3 f_\xi + \eta_3 f_\eta + \xi_1\xi_2 f_{\xi\xi} + \eta_1\eta_2 f_{\eta\eta} + (\xi_1\eta_2 + \eta_1\xi_2) f_{\xi\eta}\\ &\qquad + \tfrac{1}{6}[\xi_1^3 f_{\xi\xi\xi} + 3\xi_1^2\eta_1 f_{\xi\xi\eta} + 3\xi_1\eta_1^2 f_{\xi\eta\eta} + \eta_1^3 f_{\eta\eta\eta}] = 0\end{aligned} \tag{47}$$

where the subscripts stand for the partial derivatives. In (45)–(47) all the derivatives are taken at the point $\xi = \xi_0$, and $\eta = (q+1)\xi_0^2$. Next we introduce the values of ξ_1, ξ_2, ξ_3, η_1, η_2, η_3, ζ_1, ζ_2 and ζ_3 as functions of a_1, a_2, a_3. So (45) can be written

$$a_1 P_1 = 0 \tag{48}$$

with

$$P_1 = f_\xi + (q+2)\xi_0 f_\eta + 3(q+1)^2\xi_0^2.$$

As in the preceding case, if $P_1 = 0$, then a_1 is arbitrary without further condition. The vanishing of the coefficient of τ^2, i.e. (46), requires that

$$P_2 a_0 a_2 + P_3 a_1^2 = 0 \tag{49}$$

with

$$P_2 = -6[f_\xi + 2\xi_0 f_\eta + (2q^2 + 3q + 3)\xi_0^2] \tag{50}$$

$$\begin{aligned}P_3 = {}& 3\xi_0^2(q+1)(3q+1)(2q+3) + 3(q+1)f_\xi + 2\xi_0(q+3)(2q+1)f_\eta + 4\xi_0 q f_{\xi\xi}\\ &+ 4\xi_0^3 q(q+2)^2 f_{\eta\eta} + 8\xi_0^2 q(q+2) f_{\xi\eta}.\end{aligned} \tag{51}$$

The vanishing of the coefficient of τ^3, i.e. (47), requires that

$$P_4 a_0^2 a_3 + P_5 a_0 a_1 a_2 + P_6 a_1^3 = 0 \tag{52}$$

with

$$P_4 = -12[f_\xi - \xi_0(q-2) f_\eta + 3(q^2+1)\xi_0^2]. \tag{53}$$

P_5 is a polynomial in ξ_0 of degree three, the coefficients of which depend upon q, f_ξ, f_η, $f_{\xi\xi}$, $f_{\eta\eta}$ and $f_{\xi\eta}$ and P_6 is a polynomial in ξ_0 of degree five, the coefficients of which depend upon q, f_ξ, f_η, $f_{\xi\xi}$, $f_{\eta\eta}$, $f_{\xi\eta}$, $f_{\xi\xi\xi}$, $f_{\eta\eta\eta}$, $f_{\xi\xi\eta}$ and $f_{\xi\eta\eta}$.

Before commenting on these results we consider the relation needed to have a_r as the first non-zero coefficient, i.e.

$$x(t) = a_0 \tau^{-1/q} + a_r \tau^{r-1/q}.$$

As for the SODE we obtain an equation of the form $a_r P = 0$ which implies $P = 0$, i.e.

$$f_\xi - [(r-2)q - 2]\xi_0 f_\eta + [(r^2 - 4r + 6)q^2 - 3(r-3)q + 3]\xi_0^2 = 0. \tag{54}$$

We now return to our discussion on the possibility of having two arbitrary coefficients among a_1, a_2 and a_3. There are three possibilities

(i) a_1 and a_2 are arbitrary. Then we need $P_1 = 0$ and $P_2 = P_3 = 0$ to have respectively (48) and (49) fulfilled for any value of a_1 and a_2. It should be noted that the case $P_1 = P_2 = 0$ corresponds respectively to $r = 1$ and $r = 2$ in (54). However, we see that a further relation $P_3 = 0$ is needed. This is a compatibility condition and involves a second derivative.

(ii) a_1 and a_3 are arbitrary. We need $P_1 = 0$ and a_2 is given by (49). Now (49) is nonhomogeneous. Introducing the value of a_2 into (52) we obtain

$$P_4 a_0^2 a_3 + \left(P_6 - \frac{P_3 P_5}{P_2}\right) a_1^3 = 0$$

so that we need, in addition to $P_1 = 0$, the relations $P_4 = 0$ and $P_2 P_6 - P_3 P_5 = 0$, where $P_4 = 0$ corresponds to $r = 3$ in (54). Consequently we have three relations for a_1 and a_3 arbitrary, among them the two ARS resonances for $r = 1$ and $r = 3$. The compatibility condition $P_2 P_6 - P_3 P_5 = 0$ is now a complicated one involving up to third-order derivatives.

(iii) We want a_2 and a_3 arbitrary. Consequently $P_1 \neq 0$ and we must take $a_1 = 0$. To fulfil (49) we need $P_2 = 0$ and to fulfil (52) with $a_1 = 0$ we need only $P_4 = 0$. Hence in this last case we need only the two ARS resonance conditions and no additional one.

Now, if we suppose that the first arbitrary coefficient not equal to zero is a_{r_1} and the ARS resonance condition is fulfilled for r_2 (and of course for r_1) such that $r_1 < r_2 < 2r_1$, the equation passes the PT without further conditions. Such a result is obtained by changing τ^{r_1} in θ and noting that terms in θ^2 and θ^3 will appear in the computation in the coefficient of τ^s with $s \geqslant 2r_1$ but not in the terms in τ^{r_2} since $r_2 < 2r_1$.

For the LPS and as in the case of the RPS, we treat the first three arbitrary coefficients. We write

$$x = t^{-1/q}\left(a_0 + \frac{a_{-1}}{t} + \frac{a_{-2}}{t^2} + \frac{a_{-3}}{t^3} + \cdots\right)$$

and we turn the TODE $\zeta + f(\xi, \eta) = 0$ into an algebraic equation by expanding around the values ξ_0, $\eta_0 = (q+1)\xi_0^2$ and $\zeta_0 = (q+1)(2q+1)\xi_0^3$. Expanding $f(\xi, \eta)$ in a Taylor series and gathering the terms of same order we obtain the following four equations

$$\zeta_0 + f(\xi_0, \eta_0) = 0$$

$$\zeta_{-1} + \xi_{-1} f_\xi + \eta_{-1} f_\eta = 0$$

$$\zeta_{-2} + \xi_{-2} f_\xi + \eta_{-2} f_\eta + \tfrac{1}{2}\xi_{-1}^2 f_{\xi\xi} + \tfrac{1}{2}\eta_{-1}^2 f_{\eta\eta} + \xi_{-1}\eta_{-1} f_{\xi\eta} = 0 \tag{55}$$

$$\begin{aligned}\zeta_{-3} + \xi_{-3} f_\xi + \eta_{-3} f_\eta + \xi_{-1}\xi_{-2} f_{\xi\xi} + \eta_{-1}\eta_{-2} f_{\eta\eta} + [\xi_{-1}\eta_{-2} + \xi_{-2}\eta_{-1}] f_{\xi\eta}\\ + \tfrac{1}{6}[\xi_{-1}^3 f_{\xi\xi\xi} + 3\xi_{-1}^2\eta_{-1} f_{\xi\xi\eta} + 3\xi_{-1}\eta_{-1}^2 f_{\xi\eta\eta} + \eta_{-1}^3 f_{\eta\eta\eta} = 0.\end{aligned} \tag{56}$$

Again a great simplification is obtained from noting that

$$\xi_{-1} = \eta_{-1} = \zeta_{-1} = 0$$

indicating that a_{-1} is arbitrary. Moreover removing from the above equations all the terms involving second- and third-order derivatives, the equations (55) and (56) become

$$\zeta_{-2} + \xi_{-2} f_\xi + \eta_{-2} f_\eta = 0 \tag{57}$$

$$\zeta_{-3} + \xi_{-3} f_\xi + \eta_{-3} f_\eta = 0 \tag{58}$$

with

$$\xi_{-2} = -\tfrac{1}{2} a_0^{-2-q}[2a_0 a_{-2} - (q+1)a_{-1}^2]$$

$$\eta_{-2} = \frac{(2q+1)}{q} a_0^{-2q-2}[2a_0 a_{-2} - (q+1)a_{-1}^2]$$

$$\zeta_{-2} = \frac{-3(2q+1)(3q+1)}{2q^2} a_0^{-2-3q}[2a_0 a_{-2} - (q+1)a_{-1}^2].$$

As in the case of the LPS of a SODE the important point is that in ξ_{-2}, η_{-2} and ζ_{-2}, a_0, a_{-1} and a_{-2} provide the same factor $2a_0 a_{-2} - (q+1)a_{-1}^2$ and we have

$$2a_0^2 q\xi_0[2a_0 a_{-2} - (q+1)a_{-1}^2][f_\xi + 2(2q+1)\xi_0 f_\eta + 3(2q+1)(3q+1)\xi_0^2)] = 0. \tag{59}$$

With a_{-1} being arbitrary, if we also want a_{-2} to be arbitrary, we must cancel the bracket in (59). It is easily checked that the relation obtained is just (54) where we have taken $r = -2$. Now we must compute ξ_{-3}, η_{-3} and ζ_{-3} and we find that

$$\xi_{-3} = -a_0^{-q-3}[3a_{-1}^3 q + q^2 a_{-1}^3 + 2a_{-1}^3 - 6qa_{-1}a_{-2}a_0 - 6a_{-1}a_{-2}a_0 + 6a_0^2 a_{-3}]/3$$

$$\begin{aligned}\eta_{-3} = 2a_0^{-2q-3}[&15qa_{-3}a_0^2 + 6a_{-3}a_0^2 - 21q^2 a_{-2}a_{-1}a_0 - 18q^2 a_{-2}a_{-1}a_0\\ &-6a_{-2}a_{-1}a_0 + 8qa_{-1}^3 + 10q^2 a_{-1}^3 + 2a_{-1}^3 + 4qa_{-1}^3]/(3q)\end{aligned} \tag{60}$$

$$\begin{aligned}\zeta_{-3} = -a_0^{-3-3q}(3q+1)[&18qa_{-3}a_0^2 + 6a_{-3}a_0^2 - 24qa_{-2}a_{-1}a_0 - 24q^2 a_{-2}a_{-1}a_0\\ &-6a_{-2}a_{-1}a_0 + 6q^3 a_{-1}^3 + 9qa_{-1}^3 + 13q^2 a_{-1}^3 + 2a_{-1}^3]/q^2.\end{aligned}$$

If we want a_{-1}, a_{-2} and a_{-3} to be arbitrary (for the passing of the PT) we can see that the three values ξ_{-3}, η_{-3} and ζ_{-3} as given by (60) and introduced in (58) give three terms respectively in $a_0^3 a_{-3}$, $a_0^2 a_{-1}a_{-2}$ and $a_0 a_{-1}^3$. The coefficients in front of these terms must vanish and we obtain the three equations

$$f_\xi + (5q+2)\xi_0 f_\eta + 3(3q+1)^2\xi_0^2 = 0 \tag{61}$$

$$(q+1) f_\xi + (2q+1)(3q+2)\xi_0 f_\eta + 3(2q+1)^2(3q+1)\xi_0^2 = 0 \tag{62}$$

$$(q+2) f_\xi + 4(q+1)(2q+1)\xi_0 f_\eta + 3(2q+1)(3q+1)(3q+2)\xi_0^2 = 0. \tag{63}$$

The interesting point is that (62) and (63) are a consequence of (59) and (61), i.e. the two relations (54) with respectively $r = -2$ and $r = -3$. If we use these two equations to solve for f_ξ and f_η as functions of ξ_0 and q and introduce this solution in (62) and (63) we check that these relations are automatically fulfilled. Consequently for an LPS with a_{-1}, a_{-2}, and a_{-3} arbitrary we need only the two relations (54) for $r = -2$ and $r = -3$. □

As in the case of the SODE, no further relation is required. The above result that the coefficient a_{-1} is arbitrary is a general one and we have:

7. The NODE: resonance at $r = -1$

The resonance at $r = -1$ is a general property of the LPS of a NODE. The following proposition shows this assertion.

Proposition 4. Any NODE which takes the form

$$x^{(n)} + x^{nq+1} f\left(\frac{\dot{x}}{x^{q+1}}, \ldots, \frac{x^{(n-1)}}{x^{(n-1)q+1}}\right) = 0 \tag{64}$$

has always in the LPS the coefficient a_{-1} arbitrary.

Proof. The leading term of this self-similar ODE is in power of $t^{-1/q}$. We introduce into (64)

$$x = a_0 t^{-1/q} + a_{-1} t^{-1-1/q}.$$

The substitution of $\xi_j = \frac{x^{(j)}}{x^{jq+1}}$ $(j = 1, \ldots, n-1)$ brings (64) to the form

$$\xi_n + f(\xi_1, \ldots, \xi_{n-1}) = 0.$$

After an expansion in negative powers of t, ξ_j is

$$\xi_j = \xi_{0,j}\left[1 + \frac{a_{-1}(1 + jq - 1 - jq)}{a_0 t} + \mathrm{O}(1/t^2)\right] = \xi_{0,j}(1 + \mathrm{O}(1/t^2)).$$

We see that a_{-1} disappears from (64). This implies its arbitrariness. □

8. Reduced equations

We are interested to see what happens to the relation for the PT when, instead of the original equation, we consider the reduced equation obtained by taking into account the time translation symmetry. This point is specially interesting since in that case x becomes the independent variable and $\dot{x} = y$ the dependent one. The singularity at time t_0 (x going to infinity) obtained in the RPS will correspond in the reduced equation to $x \to \infty$ and consequently to an LPS.

8.1. *Case of the SODE*

Taking x as a new variable and $y = \dot{x}$ as a new function we write the SODE (5) as

$$y\frac{\mathrm{d}y}{\mathrm{d}x} + x^{2q+1} f(yx^{-q-1}) = 0.$$

The leading term is now $y = \xi_0 x^{q+1}$, where ξ_0 is given by the same relation as before, namely $f(\xi_0) + (q+1)\xi_0^2 = 0$. We construct an LPS with $y = \xi_0 x^{q+1} + a_r x^{q+r+1}$. To obtain an arbitrary a_r (with $r < 0$) we must satisfy

$$f_\xi(\xi_0) + [2(q+1) + r]\xi_0 = 0.$$

Again, if this relation is not fulfilled, we must take $a_r = 0$. Comparing with the relation (24) obtained for the RPS of the original equation we see that we just have to replace qr by $-r$. If we remember that in the RPS r is positive and in the LPS negative, we see that for the same f and with $q \neq 1$ the reduced equation is q times 'richer' than the original one. This is not surprising since the leading term in $\tau^{-1/q}$ of the original equation has q branches while the reduced equation is analytic. In section 10 we make a comment, by means of an example, on the significance of this difference.

8.2. Case of the TODE

The above property for the SODE will also be found in the TODE case. In the same way, by introducing $y = \dot{x}$ in equation (6), we find that

$$y^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + x^{3q+1} f\left(\frac{y}{x^{q+1}}, \frac{y}{x^{2q+1}}\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0.$$

The leading term is still given by $y = \xi_0 x^{q+1}$, where ξ_0 is defined by the relation (44). For the coefficient a_r to be arbitrary we must perform an LPS with

$$y = \xi_0 x^{q+1} + a_r x^{r+q+1} \qquad (r < 0)$$

and the following relation must be satisfied

$$f_\xi(\xi_0, \eta_0) + (2q + 2 + r) f_\eta(\xi_0, \eta_0) + [6q^2 + (4r + 9)q + r(r + 3) + 3]\xi_0^2 = 0. \tag{65}$$

If it is not satisfied, then $a_r = 0$. We can immediately note that replacing qr by $-r$ $(r < 0)$ in relation (54) (which has been obtained in the building of the RPS of the original equation) we obtain the relation (65) above. Finally the construction of an LPS for the reduced equation transforms the multivalued function

$$x(t) = t^{-1/q} \sum_{j \geqslant 0} a_j (t - t_0)^j$$

into a univalued function as

$$y(x) = x^{q+1} \sum_{j \geqslant 0} a_{-j} x^{-j}$$

which is holomorphic. It may well be that the holomorphy is responsible for the fact that the reduced equation describes a situation q times 'richer' from the point of view of integrability. Moreover, in a geometrical interpretation, the reduced equation allows us to work on the Riemann surfaces of the multivalued function, the leading term of which being $t^{-1/q}$. For the second function the Riemann surface is represented by q superposed sheets which join at the critical points $t = t_0$ and $t = \infty$.

9. Algebraic nature of the equation for the first integral of the SODE

The equation (8) can be written

$$\frac{\mathrm{d}x}{x} + \frac{\xi}{F(\xi)} = 0 \tag{66}$$

with

$$F(\xi) = f(\xi) + (q + 1)\xi^2.$$

Let us suppose that $F(\xi)$ is an analytic function. Then $\frac{\xi}{F(\xi)}$ can be expressed through the poles ξ_{0i} of $F(\xi)$

$$\frac{\xi}{F(\xi)} = \sum_{i \in \Omega} \frac{\xi_{0i}}{F_\xi(\xi_{0i})(\xi - \xi_{0i})} \tag{67}$$

where Ω is a finite set and

$$F_\xi(\xi_{0i}) = f_\xi(\xi_{0i}) + 2(q + 1)\xi_{0i}. \tag{68}$$

The ξ_{0i} are the poles giving the dominant term. To obtain a resonance r we must have the relation (24) fulfilled, i.e.

$$f_\xi(\xi_{0i}) + 2(q + 1)\xi_{0i} = q r_i \xi_{0i}. \tag{69}$$

Equations (68) and (69) show that $F_\xi(\xi_{0i}) = qr_i\xi_{0i}$. Substituting this into (67) we obtain

$$\frac{\xi}{F_\xi(\xi)} = \sum_i \frac{1}{qr_i(\xi - \xi_{0i})}$$

and (66) can be integrated to give the constant of motion

$$I = x\prod_i(\xi - \xi_{0i})^{\frac{1}{q_i r_i}} \tag{70}$$

which can be turned into a polynomial expression since q and r_i are integers. The crucial point is the disappearance of ξ_{0i} to obtain an equation involving only the different r_i and, of course, q. In the general case we must check the nature of the value r_i connected to the pole ξ_{0i}. For example, let us consider again the case of the SODE with $f(\xi) = \lambda\xi^2 + \xi + \nu$ the asymptotic form of which we sought in section 2. The two roots ξ_{01} and ξ_{02} are now given by the solution of

$$(q + \lambda + 1)\xi_0^2 + \xi_0 + \nu = 0. \tag{71}$$

A first resonance at r in an RPS gives

$$\xi_{01} = \frac{1}{qr - 2(q + \lambda + 1)} \tag{72}$$

corresponding to $\nu = [q(1-r) + \lambda + 1]/[qr - 2(q + \lambda + 1)]^2$. We obtain the second root

$$\xi_{02} = \frac{(1-r)q + \lambda + 1}{(q + \lambda + 1)[qr - 2(q + \lambda + 1)]} \tag{73}$$

and, with (67), the first integral (70) can now be written

$$I = x^{\lambda qr}[\dot{x} - \xi_{01}x^{q+1}]^{q+\lambda+1}[\dot{x} - \xi_{02}x^{q+1}]^{(r-1)q-\lambda-1}. \tag{74}$$

Here the polynomial nature of the first integral I requires that λ be an integer.

10. Characteristic example for the SODE

In equation (5) the case where

$$q = 1 \qquad \text{and} \qquad f(\xi) = \xi + k \tag{75}$$

arises in applications such as fusion of pellets [5] and astrophysics [14] and has received extensive mathematical treatment [13]. The equation (5) becomes

$$\ddot{x} + x^q\dot{x} + kx^{2q+1} = 0 \tag{76}$$

the reduced form of which being

$$y\frac{\mathrm{d}y}{\mathrm{d}x} + x^q y + kx^{2q+1} = 0. \tag{77}$$

We introduce this special form for two reasons. The first is because we wish to extend the classical PT for which, when applied to the case $q = 1$, only the values $k = \frac{1}{9}$, 0 and -1 are considered as corresponding to the indicial $r = 1, 2, 3$ respectively. The second is because we wish to explore the implications for other possible indicial resonances. The answer is found by looking at the first integral as defined by (8) for the values of k when $f(\xi)$ is replaced by (75). Moreover we use (21) with $f(\xi_0) = k - q^{-1}a_0^{-q}$ and the resonance index relation (24) where $f_\xi(\xi_0) = 1$ to obtain

$$k = \frac{q + 1 - qr}{(2 + 2q - qr)^2}. \tag{78}$$

Thus the first integral (8) becomes

$$I = (\dot{x} - \xi_+ x^{q+1})^{q+1} (\dot{x} - \xi_- x^{q+1})^{-1-q+qr} \tag{79}$$

where ξ_+ and ξ_- are the roots of $(q+1)\xi^2 + \xi + k = 0$, which with (78) take the forms

$$\begin{aligned} \xi_+ &= \frac{1}{qr - 2q - 2} \\ \xi_- &= \frac{q + 1 - qr}{(q+1)(qr - 2q - 2)}. \end{aligned} \tag{80}$$

The first integral (79) is a polynomial function for all values of integer q and r. When k is selected such that the equation passes the PT, this result is in fact general for all SODEs of this form as was demonstrated in section 9. The relation (79) can be considered as an equation giving $\dot{x}$ as a function of x, but the performance of the inversion implies that this equation should be of degree less than five and consequently cannot be given an explicit global form except for

$$\begin{array}{lll} q = 1 & r = 1, 2, 3 & (k = \frac{1}{9}, 0, -1) \\ q = 2 & r = 1, 2 & (k = \frac{1}{16}, -\frac{1}{4}) \\ q = 3 & r = 1 & (k = \frac{1}{25}). \end{array}$$

Note that $q = 1$ and $r = 4$ give a fourth-degree equation in $\dot{x}$, but must be rejected because k is no longer finite. We emphasize that these constraints are only on the existence of an explicitly global representation of $\dot{x}$ as a function of x. The inverse function theorem guarantees the invertibility for all values of q and r. The cases with $q = 2$ and $q = 3$ are new.

The passing of the PT for $r = 1$ and an arbitrary q deserves a special treatment. It was noted [10] that for $q = 1$ and $k = \frac{1}{9}$ the change of dependent variable

$$x = 3\dot{u}/u \tag{81}$$

gives the simplest possible equation namely $\dddot{u} = 0$. It turns out that this result can be generalized for an arbitrary q with the value of k corresponding to the resonance $r = 1$ i.e. accordingly to (78) $k = 1/(q+2)^2$. Let us introduce the change of variable

$$x = \left(1 + \frac{2}{q}\right)^{1/q} \left(\frac{\dot{u}}{u}\right)^{1/q} \tag{82}$$

which generalizes (81). The equation for u becomes

$$\dot{u}\,\dddot{u} = \left(1 - \frac{1}{q}\right)\ddot{u}^2 + \frac{1}{q}\frac{\dot{u}^4}{u^2}[1 - k(q+2)^2]. \tag{83}$$

Since $k = 1/(q+2)^2$ the last term in the r.h.s. of (83) cancels and we get

$$\dot{u}\,\dddot{u} = \left(1 - \frac{1}{q}\right)\ddot{u}^2. \tag{84}$$

However, (84) now has three symmetries. First the invariance with respect to time translation and the two homogeneity symmetries characterized by $t\partial/\partial t$ and $u\partial/\partial u$. After some calculations we obtain for $x(t)$

$$x(t) = \left[\frac{(q+1)(q+2)}{q}\right]^{1/q} \left[\frac{A(At + B)^q}{(At + B)^{q+1} + C}\right]^{1/q}. \tag{85}$$

The constants A, B and C in (84) are determined by the initial conditions. In fact dividing B by A and C by A^{q+1} we see that we have only two constants to be determined and

Table 1. Values taken by the second resonance r_2.

	r_1					
q	1	2	3	4	5	6
1	$^*-2$	$^*\infty$	^{*6}	4	$\frac{10}{3}$	3
2	$^*-3$	^{*6}	3	$\frac{12}{5}$	$\frac{15}{7}$	2
3	$^*-4$	4	$\frac{12}{5}$	2	$\frac{20}{11}$	$\frac{12}{7}$
4	-5	$\frac{10}{3}$	$\frac{15}{7}$	$\frac{20}{11}$	$\frac{5}{3}$	$\frac{30}{19}$
5	-6	3	2	$\frac{12}{7}$	$\frac{30}{19}$	$\frac{3}{2}$

without any loss of generality we can take $A = 1$. Then B and C are determined by the two initial conditions $x(0)$ and $\dot{x}(0)$. We see that the simple integrable case pointed out by the resonance $r = 1$ in the case $q = 1$ can be generalized to any value of q.

In that case we note that (79) becomes an equation of degree greater than four if $q \geqslant 4$. In our opinion it shows simply that many roads leading to integrability exist for a function having passed the PT, the existence of a polynomial relation connecting $\dot{x}$ and x being one and the case q arbitrary and $r = 1$ being another. We will come back to that point.

For completeness, let us show finally how we recover (79) from (70). As given by (78), k is selected to give a resonance at an integer r_1. Introducing this value of k in (21) we obtain two values for ξ_0. One, let us say ξ_{01}, introduced in (24) gives a resonance at r_1, which is quite normal since this value of k was built from r_1. The other using (24) gives a value r_2

$$r_2 = \frac{r_1(q+1)}{(r_1-1)q-1} \tag{86}$$

and (70) can be written

$$I = x(\xi - \xi_{01})^{\frac{1}{qr_1}}(\xi - \xi_{02})^{\frac{(r_1-1)q-1}{r_1 q(q+1)}}. \tag{87}$$

Taking the power $r_1 q(q+1)$ of the two members of (87) and remembering that $\xi = \dot{x}/x^{q+1}$ we recover (79) and the polynomial nature of the equation connecting $\dot{x}$ and x is proved.

Now we come back to an important point dealing with the need to have the PT fulfilled for all the singularities of the equation. (85) provides the answer for the SODE (76). If r_1 is an integer r_2 is at least a rational number and the second singularity exhibits at least a weak PT with the possibility of writing the invariant in a polynomial form. Table 1 is interesting: it gives for a resonance at r_1 the value r_2 of the other resonance for different values of q.

The values where we found a possibility of integration by noting that $\dot{x}$ can be solved as a function of x are indicated by an asterix. For all these values both r_1 and r_2 are integer. But other cases point out possible integrability with two integer values for r_1 and r_2. First, for any value of q, the existence of a LPS with resonances at $r_1 = 1$ and $r_2 = -(q+1)$ as indicated by the first column of table 1. The solution has just been given in (85). But other cases are possible. For example $q = 5$ with $r_1 = 2$ and $r_2 = 3$ and $q = 3$, $r_1 = 2$, $r_2 = 4$. For these cases the integration is still to be done.

Note that for $q = 1$ and $r = 2$ the second resonance is infinite, $k = 0$ and the equation becomes $\ddot{x} + x\dot{x} = 0$ with the obvious first integral $\dot{x} + x^2/2 = I$ in agreement with (79) and (80).

Returning to the reduced equation (77) we can show that for $q > 1$ explicitly new solvable cases exist because we can replace qr by $-r$ in (78)–(80). These are worthy of separate investigation. They lead to quadratures where the integrands are explicitly known in the cases

$$q = 2 \qquad \text{with } r = -1, -2, -3, -4 \qquad (k = \tfrac{2}{25}, \tfrac{1}{16}, 0, -\tfrac{1}{4})$$
$$q = 3 \qquad \text{with } r = -1, -2, -3, -4 \qquad (k = \tfrac{3}{49}, \tfrac{1}{18}, \tfrac{1}{25}, 0)$$

in addition to the case $q = 1$ with the values $r = -1, -2, -3$. Consequently, we find other new cases of integration, namely the k values corresponding to $r = -1, -3$ in the case of $q = 2$ and $r = -1, -2, -4$ in the case of $q = 3$. Note that the k values for which the SODE passes the PT are a subset of the k values for which the reduced first-order equation passes the test. It appears that the reduced equation is 'richer', but not automatically q times 'richer' since the requirement for the $\dot{x}$ equation to be of degree less than five introduces constraints on the possible values of q and r.

11. Characteristic example for the TODE

The case of the generalized third-order equation, (6), may be regarded as a generalized Chazy equation [4]. It is the Chazy equation when

$$f(\xi, \eta) = k\xi^2 + \eta \tag{88}$$

and $q = 1$ and $k = -\frac{3}{2}$. When $k = -\frac{11}{7}$, it is known as Bureau's equation [3]. Both third-order equations were treated by Fordy and Pickering [6].

Equation (6) can always be reduced to the first-order equation

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \frac{f(\xi, \eta) + (2q + 1)\xi\eta}{(q + 1)\xi^2 - \eta} \tag{89}$$

where ξ and η are as defined above. However, there is no longer the equivalent of reduction to quadratures through the first integral (89) since there are insufficient symmetries. Nevertheless, there are cases, when f is given by (88), for which it is integrable. In the case of the Chazy equation this follows from the existence of a third-point symmetry, namely

$$G_3 = t^2 \frac{\partial}{\partial t} + 2(6 - xt)\frac{\partial}{\partial x}.$$

This is not the case for Bureau's equation, but when $k = 1$, the equation is integrable.

Clearly (89) can be reduced to a quadrature only under appropriate restrictions on f. With (88) in (6), the equation becomes

$$\dddot{x} + x^q \ddot{x} + kx^{q-1}\dot{x}^2 = 0. \tag{90}$$

We consider (90) hereafter adopting the general procedure described in section 4. The explicit building of the series is informative in terms of interpretation of the somewhat different natures of the RPS and LPS.

Now, with $f(\xi, \eta) = k\xi^2 + \eta$, we have $f(\xi_0, (q + 1)\xi_0^2) = (q + 1 + k)\xi_0^2$; $f_\xi = 2k\xi_0$ and $f_\eta = 1$. This gives the value ξ_0 and k namely

$$\xi_0 = -\frac{q + 1 + k}{(q + 1)(2q + 1)}$$
$$k = -\frac{(q + 1)[q^2r^2 - 2\,(q + 1)qr + (q + 1)(2q + 1)]}{q^2r^2 - q(4q + 3)r + (q + 1)(2q + 1)}$$

which for $q = 1$ becomes

$$k = -2\frac{r^2 - 4r + 6}{(r - 1)(r - 6)}. \tag{91}$$

If we seek an RPS for the generalized Chazy equation, we should look for values of r where we need only the two ARS resonance relations: (54) for example with $r = 2, 3$. We find indeed that $k = 1$ and $q = 1$ allow (54) to be satisfied for these two values. More surprising is the existence of an LPS with a_{-1}, a_{-2} and a_{-3} arbitrary for $q = 1$ and $k = -\frac{3}{2}$. Indeed we check that (59) and (61) (i.e. (54) for $r = -2$ and $r = -3$) are fulfilled. Alternatively we can check that $r = -2$ and $r = -3$ introduced in (91) give the same value $k = -\frac{3}{2}$, while $r = 2$ and $r = 3$ gives $k = 1$. The fulfilment of (59) and (61) in that last case and the first case, ($k = q = 1$) were pointed out by Fordy and Pickering [6].

Other characteristic examples for this TODE are those which satisfy the property

$$k = q$$

for which (75) gives $\xi_0 = -(q + 1)^{-1}$ and (54) gives the resonances

$$r_1 = \frac{q + 1}{q} \qquad r_2 = \frac{2q + 1}{q}.$$

We note that when $q = k = 1/m$ with m an integer number these resonances are $r_1 = m+1$ and $r_2 = m + 2$ and thus satisfy the property $r_1 < r_2 < 2r_1$ with $r_1 > 1$. Consequently the PT is passed.

Of course for $m = 1 = k = q$ the result $r = 2, 3$ is recovered. At this point one may wonder why $k = q$ is so interesting. An explanation can be given, noting that for $k = q$ (89) admits a solution $\eta = -\xi$. Then we go back to the original equation (90) which becomes

$$\dddot{x} + x^q \ddot{x} + q x^{q-1} \dot{x}^2 = 0. \tag{92}$$

We can proceed to a first integration obtaining

$$\ddot{x} + x^q \dot{x} = A$$

and on integrating again

$$\dot{x} + \left(\frac{x^{q+1}}{q + 1}\right) = At + B. \tag{93}$$

This last equation can be proved to be integrable for certain values of q. The case $A = 0$ corresponds to initial conditions where

$$\ddot{x}(0) + x^q(0)\dot{x}(0) = 0$$

i.e. precisely

$$\eta(0) + \xi(0) = 0.$$

For this class of initial conditions the equation is integrable.

12. Conclusion

In contradistinction to other work we emphasize the building of an explicit Laurent series. The classical Painlevé series is based on an expansion in a disk centred on a movable singularity at $t = t_0$. In that case we need $n - 1$ additional arbitrary coefficients for an equation of order n. On the other hand we can build another series outside the disk containing t_0 and this is an expansion for $t \mapsto \infty$. In this case we need n arbitrary coefficients. Given the existence of the correct number of resonances there remains the need for consistency. In this paper we have studied particularly the properties of the second-and third-order equations possessing the symmetries of time translation and similarity. For the family of second-order equations we always obtain a first integral I and have shown in the special case of (76) that this first integral is a polynomial in $\dot{x}$ for the values of the parameters leading to the passing of the PT. A generalization of this result is possible and was given in section 9. This polynomial provides an algebraic equation in $\dot{x}$ as a function of x and I. To obtain the final quadrature in global form this polynomial must be of degree less than five and this gives, among others, the three classical values of the parameter k, namely $\frac{1}{9}$, 0, and -1. It is interesting to observe that the problems of multivaluedness associated with the second-order equation (5) for $q > 1$ disappear when we use the time translation symmetry to reduce (5) to first order. This first-order equation must be studied with an LPS since the new independent variable x goes to infinity and corresponds to the movable singularity at $t = t_0$. This reminds us that the passing of the PT is representation dependent. We also give an explicit proof for the nth order equations possessing time translation and self-similarity symmetries that they always have an arbitrary a_{-1} coefficient in the construction of the LPS. This result gives more pedagogical sense to the arbitrariness of a_{-1} than usual arguments presented in the literature on the meaning of the resonance $r = -1$. New integrable cases have been obtained together with old ones, as characteristic examples of the second- and third-order equations.

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